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On the evolution of disturbances at an inviscid interface

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The initial-value problem for the evolution of the interface $\eta(x,t)$ separating two unbounded, inviscid streams is considered in the framework of linearized analysis. Given the initial shape $y = \epsilon\eta_0(x)$ of the interface at $t = 0$ the objective is to calculate the interface shape $\eta(x,t)$ for later times. First, it is shown that, if the vortex sheet is of infinite extent, if surface tension is absent and if the two streams are of the same density, the evolution is given by

$$\eta(x,t) = \epsilon(1-\alpha)^{-1}\,\mathrm{Re}\,[\{(1-\alpha)+(1+\alpha)\,i\}\,\eta_0\{x-\tfrac{1}{2}((1+\alpha)+(1-\alpha)\,i)\,t\}],$$

where α ($\neq 1$) is the ratio of the speeds of the streams, provided the initial interface shape $\epsilon\eta_0(x)$ is analytic and its Fourier transform decays sufficiently rapidly. An interesting consequence is that it is possible, under certain circumstances, for the interface to develop singularities after a finite time. Next it is shown that when the two streams move at the same speed ($\alpha = 1$) the growth of η is given by

$$\eta(x,t) = \epsilon\eta_0(x-t) + \epsilon t\, d\eta_0(x-t)/dx$$

with mild restrictions on $\eta_0(x)$. The major effect of surface tension, it is found, is to prevent the occurrence of singularities after a finite time, a distinct possibility in its absence. Finally the vortex sheet shed by a semi-infinite flat plate is considered. The unsteady mixed boundary-value problem is formally solved by using parabolic co-ordinates and Fourier–Laplace transforms.

1. Introduction

Consider two inviscid, incompressible fluid streams of infinite extent in uniform parallel motion, one on top of the other, with the interface $\eta(x,t)$ initially close to the plane $y = 0$ (see figure 1). Let α and β be the ratios of their speeds and densities respectively, i.e. $\alpha = U_2/U_1$ and $\beta = \rho_2/\rho_1$. Gravity is assumed to act in the negative y direction. We study here the evolution of the interface $\eta(x,t)$ given the initial interface shape $\epsilon\eta_0(x)$ particularly when the interface is Helmholtz unstable. The analysis is throughout in the framework of linear theory. The infinite vortex sheet is considered first, following which the semi-infinite vortex sheet, i.e. that shed by a semi-infinite flat plate, is considered.

The study of the initial-value problem for surface wave motions is classical; indeed, Lamb (1932) describes in detail the evolution of surface waves from a localized initial disturbance and there is now a considerable body of work on the initial-value problem when the motions considered are classically stable. On the other hand, for fluid motions that are basically unstable the corresponding literature appears to be far less extensive. Case (1962) and Birkhoff (1962) have discussed the role of the initial-value

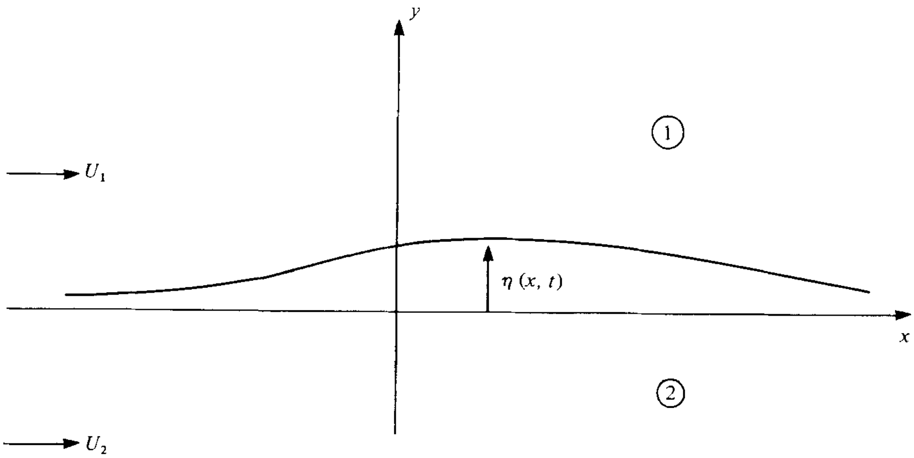

FIGURE 1. The interface between two uniform, inviscid streams.

problem in the study of hydrodynamic stability. Case has pointed out that, while modal analysis is indeed useful for physical understanding, the evolution problem needs to be studied as individual modal growth rates may not give a true picture of the evolution of an actual initial disturbance. Moreover, an initial-value problem solved by operational methods is unlikely to miss the continuous spectrum, if it exists, as a modal analysis might.

Work related to the present one has been done in the acoustic context by Miles (1958), Jones (see, for example, Jones 1972; Jones & Morgan 1972; and Jones 1978 and other references quoted therein) and Crighton & Leppington (1974). For the infinite vortex sheet, the present work complements that of Miles in that an exact solution is given for the incompressible case. The concern in the case of the semi-infinite vortex sheet, in the acoustic context, is primarily with the possible enhancement of the acoustic radiation by the instability of an initially plane vortex sheet and by interactions at the trailing edge of the plate. In the incompressible situation considered here the instability is not triggered externally, as in the acoustic case, but by the non-planar nature of the initial interface shape. It is to be pointed out that the results obtained here do not follow from those of the acoustic case by a limiting process. Moreover, it is to be noted that in the acoustic case, where the initial-value problem has been considered, the initial interface shape has, naturally, been assumed to be plane; on the other hand, in the incompressible case it is the initial undulation which leads to interface growth. A feature of some interest is that the incompressible problem permits solution by elementary means without recourse to the Wiener–Hopf technique.

2. Formulation

Let α and β be the ratios of the speeds and densities respectively of the two streams. Let all lengths be normalized by U_1^2/g, time by U_1/g and speeds by U_1. Then the normalized perturbation potentials $\phi_1(x,y,t)$ and $\phi_2(x,y,t)$ valid in regions 1 and 2 satisfy

$$\nabla^2\phi_1 = \nabla^2\phi_2 = 0 \tag{1}$$

in their respective regions of validity. Since we require the perturbation velocities to decay far from the interface, the boundary conditions for $|y| \to \infty$ are

$$|\nabla\phi_1| \to 0 \quad \text{for} \quad y \to \infty, \quad |\nabla\phi_2| \to 0 \quad \text{for} \quad y \to -\infty. \tag{2}$$

The kinematic conditions for the continuity of the interface and the pressure across it require the linearized boundary conditions (for $t > 0$)

$$\left.\begin{aligned} \eta_t + \eta_x &= \phi_{1y} \qquad (3.1)\\ \eta_t + \alpha\eta_x &= \phi_{2y} \qquad (3.2)\\ \phi_{1t} + \phi_{1x} + \eta &= \beta\{\phi_{2t} + \alpha\phi_{2x} + \eta\} \qquad (3.3) \end{aligned}\right\} \quad \text{on} \quad y = 0, \quad x > 0.$$

Note that in the above we have ignored surface tension, whose effects will be considered in § 3.3. For the infinite vortex sheet (3) also hold on $y = 0$, $x \leqslant 0$. On the other hand for the vortex sheet shed by a flat plate, the rigid plate requires that the normal components of velocity vanish on it:

$$\left.\begin{aligned} \eta(x,t) = 0, \quad x \leqslant 0; \\ \phi_{1y} = 0, \quad \phi_{2y} = 0 \quad \text{on} \quad y = 0, \quad x \leqslant 0. \end{aligned}\right\} \tag{4}$$

Now for initial conditions we assume that $\eta(x, 0) = \epsilon\eta_0(x)$ ($\epsilon \ll 1$) is prescribed and that the potentials $\phi_1(x, y, 0)$ and $\phi_2(x, y, 0)$ correspond to steady flow along a hard boundary given by $\epsilon\eta_0(x)$. Physically the problem amounts to determining the shape of the interface at later times if the thin wall separating the two streams were suddenly to dissolve or disappear through the side wall of a wind or water tunnel.

For the infinite vortex sheet the formulation is now complete. For the semi-infinite vortex sheet one possibly needs further to specify a condition at the trailing edge of the plate to ensure uniqueness. Orszag & Crow (1970) consider what they call a 'rectified Kutta condition' and a 'full Kutta condition'. It appears to us that the rectified Kutta condition, where a steady (time independent) solution is added to the modal solution, has little basis; in fact in the initial-value problem it certainly cannot be incorporated, even if one wished to do so. The full Kutta condition, requiring the flow to leave the trailing edge tangentially, seems quite reasonable but the artifice that was possible in the modal analysis does not appear possible here in the initial-value problem. We observe that the usual source of non-uniqueness in conventional aerofoil theory, the multiple-connectedness of the domain, is absent in the present problem. The non-uniqueness here, if any, is a result of the geometry of the boundary, i.e. the sharp trailing edge (cf. non-uniqueness in the solution for flow past a sharp corner). This being the case, we shall only require that the solution be the one least singular at the trailing edge. It is recognized that the trailing-edge condition is a source of difficulty, but it appears that only a detailed viscous analysis might possibly shed light on the issue.

3. The infinite vortex sheet

3.1. *The formal solution*

Since the method of solution for the problem formulated in § 2 for the infinite vortex sheet is standard we only outline the derivation here.

On (*a*) Fourier transforming the governing equations for the potentials in the two regions, (*b*) applying the boundary conditions for $|y| \to \infty$, and (*c*) solving the differential equations in time resulting from the boundary conditions at the interface, one finds that the Fourier transform $\tilde{\eta}(k,t)$ of $\eta(x,t)$, provided α and β are not both unity, is given by

$$\tilde{\eta}(k,t) = \tfrac{1}{2}\epsilon[\{1-ik(1+\alpha\beta)(k^2(1-\alpha)^2\beta-(\beta^2-1)|k|)^{-\frac{1}{2}}\}e^{p_1(k)t} + \{1+ik(1+\alpha\beta)(k^2(1-\alpha)^2\beta-(\beta^2-1)|k|)^{-\frac{1}{2}}\}e^{p_2(k)t}]\,\tilde{\eta}_0(k), \tag{5.1}$$

where $\tilde{\eta}_0(k)$ is the Fourier transform of $\eta_0(x)$ and the coefficients in the exponents are given by

$$p_{1,2}(k) = \{ik(1+\alpha\beta) \pm (k^2(1-\alpha)^2\beta-(\beta^2-1)|k|)^{\frac{1}{2}}\}(\beta+1)^{-1}. \tag{5.2}$$

If $\alpha = \beta = 1$ the transform is simply given by

$$\tilde{\eta}(k,t) = \epsilon\, e^{ikt}(1-ikt)\,\tilde{\eta}_0(k). \tag{5.3}$$

The interface shape is then given by the inversion formula

$$\eta(x,t) = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty}\tilde{\eta}(k,t)\,e^{-ikx}\,dk \tag{6}$$

provided the integral exists. One observes that the classical dispersion relation is contained in formula (5.2).

We now consider sufficient conditions under which the integral (6) exists and hence constitutes a solution to the initial-value problem. First, if the two streams move at the same speed ($\alpha = 1$) and the lower fluid is denser than the upper one then $p_{1,2}$ are imaginary for all k and the integral converges under the mild restriction that $\tilde{\eta}_0(k)$ be absolutely integrable. This is the usual case considered, corresponding to all modes being stable. Second, if $\alpha \neq 1$, $\mathrm{Re}\{p_{1,2}\} \sim O(k)$ as $k \to \infty$ and the integral will exist for $t > 0$ only if $\tilde{\eta}_0(k)$ decays sufficiently rapidly for $|k| \to \infty$. For example, a sufficient condition is that $\tilde{\eta}_0(k) \sim k^n \exp(-ak^2)$ for $|k| \to \infty$ with $a > 0$. An interesting possibility is for the solution to exist only for a finite time interval. Say

$$\tilde{\eta}_0(k) \sim k^n \exp(-a|k|)$$

as $|k| \to \infty$, $a > 0$; the solution will exist only for some finite time interval $0 \leqslant t < T$, where T can be easily determined. This phenomenon is connected with the fact, pointed out by Birkhoff (1962), that the perturbation problem is 'not mathematically well set in the sense of Hadamard'; Birkhoff also points out that the inclusion of the effects of surface tension or viscosity would eliminate this difficulty. It is interesting to note that similar difficulties arise in the compressible case. For example Jones & Morgan (1972) found that exponentially growing (Helmholtz unstable) solutions had to be retained to satisfy causality. It is likely, therefore, that even in the compressible case singularities may occur after a finite time in the absence of surface tension or viscosity.

There are two special cases of considerable interest for which explicit formulae are obtainable. If the streams are of the same density ($\beta = 1$) but move at different speeds ($\alpha \neq 1$) and if the Fourier integral exists, (6) can be immediately integrated, using the shift theorem, to give

$$\eta(x,t) = \epsilon(1-\alpha)^{-1}\,\mathrm{Re}\,[\{(1-\alpha)+(1+\alpha)\,i\}\,\eta_0\{x-\tfrac{1}{2}[(1+\alpha)+(1-\alpha)\,i]\,t\}], \tag{7}$$

where $\eta_0(x)$ has been assumed to be analytic. On the other hand, if the streams are of the same density and move at the same speed (i.e. $\alpha = \beta = 1$), it is clear from (5.3) that we need only require that $\eta_0(x)$ and $d\eta_0/dx$ be absolutely integrable for the integral (6) to exist. Under these mild restrictions we have

$$\eta(x,t) = \epsilon\eta_0(x-t) + \epsilon t\, d\eta_0(x-t)/dx. \tag{8}$$

Thus, in this case the growth is linear in time,† as predicted by modal analysis, and proportional to the local gradient. We observe that (8) could be obtained by taking the limit $\alpha \to 1$ in (7), but the conditions would then be more restrictive than necessary. Further, (8) implies a loss of smoothness in general, as $\eta(x,t)$ now also depends on the initial slope.

It is to be noted that there is no difficulty in physically interpreting the case $\alpha = \beta = 1$ in the context of the initial-value problem as posed. For example this would correspond to the situation where the thin wall separating two regions of the same fluid is suddenly removed through the side wall of a wind tunnel. In fact Lord Rayleigh (see Lamb 1932) attributed the flapping of flags and sails to this instability. However, as has been pointed out by Birkhoff (1962), the phenomena under question are likely to be greatly influenced by the density of the cloth and the boundary layers on either side.

3.2. *Examples*

A few simple examples are now presented to illustrate the results of the last subsection. Assume both streams to be of the same density.

Consider the initial profile given by $\eta_0(x) = \exp(-x^2)$. Since $\tilde{\eta}_0(k) \sim \exp(-\frac{1}{4}k^2)$, formula (7) applies and one has

$$\eta(x,t) = \epsilon(1-\alpha)^{-1}[(1-\alpha)\cos v - (1+\alpha)\sin v]\exp\{-(x-\alpha t)(x-t)\},$$

where $v = (1-\alpha)t(x-\frac{1}{2}(1+\alpha)t)$. At any given time the amplitude is bounded and its maximum value is of the order of $\exp\{\frac{1}{2}(1-\alpha)^2t^2\}$, i.e. the growth is even more rapid than might be suggested by the modal solutions, each of which grow exponentially. One observes also that the peak of the disturbance travels at a speed approximately equal to the mean of the speeds of the two streams, $\frac{1}{2}(1+\alpha)$. Figure 2 shows the growth of the interface for $\alpha = 0$ and $\alpha = 0{\cdot}5$.

A considerably different growth is evinced by the profile

$$\eta_0(x) = (1+x^2)^{-1}, \quad \tilde{\eta}_0(k) \sim \exp(-|k|).$$

In order to simplify the algebra assume counter-flowing streams with $\alpha = -1$. Since by (5.2) $p_{1,2} = \pm k$ it is clear that the integral (6) exists for $0 \leqslant t < 1$. During this time interval formula (7) holds and so

$$\eta(x,t) = \mathrm{Re}\,[\epsilon\eta_0(x-it)] = \frac{\epsilon}{2}\left[\frac{1-t}{(1-t)^2+x^2} + \frac{1+t}{(1+t)^2+x^2}\right].$$

† A referee has kindly given a physical explanation for the result (8). 'When the streams have the same speed no energy is fed into the disturbance, but since the densities are the same, there is no restoring force either. Hence when the hypothetical barrier separating the streams is removed at $t = 0$ the fluid elements at the interface move in straight trajectories with their instantaneous velocity at $t = 0$, $d\eta_0(x-t)/dx$, i.e. the interface displacement grows linearly with time.'

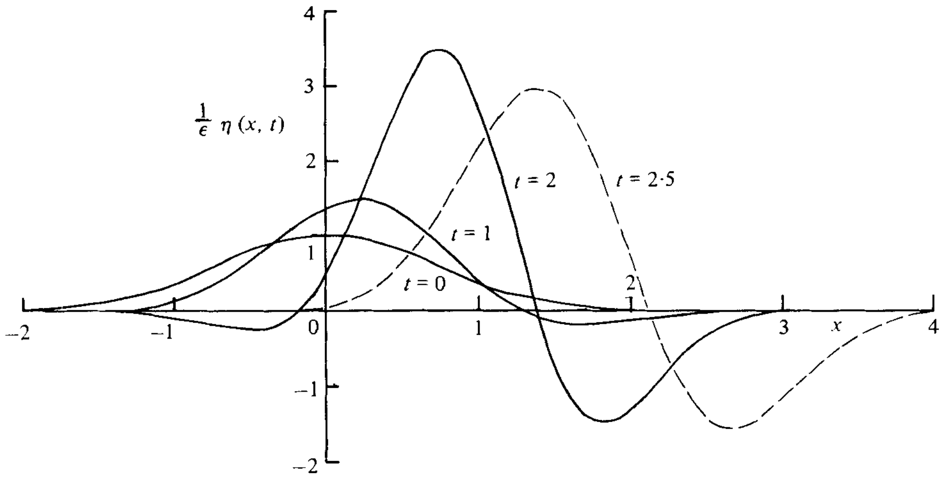

FIGURE 2. The growth of the interface when $\beta = 1$ and $\eta_0(x) = \exp(-x^2)$. ——, $\alpha = 0$; - - - -, $\alpha = 0{\cdot}5$.

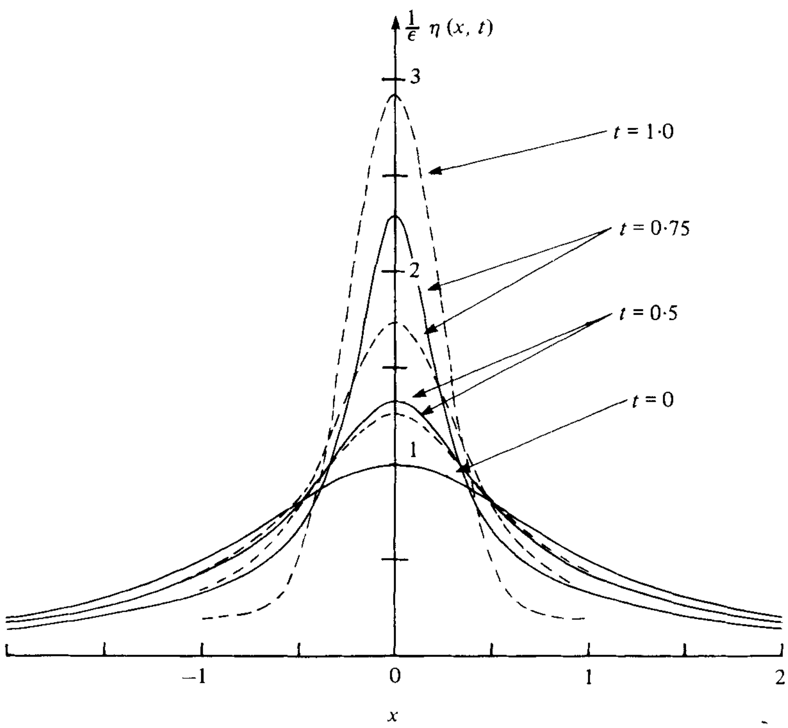

FIGURE 3. The evolution of the interface when $\beta = 1$, $\alpha = -1$ and $\eta_0(x) = (1+x^2)$. ——, no surface tension; - - - -, with surface tension, $\sigma = 0{\cdot}1$.

Since the streams are in counter-flow at the same speed the disturbance does not move but just grows in time, becoming unbounded at $x = 0$ as $t \to 1$ (see figure 3). The solution does not exist for $t \geqslant 1$. We note that the solution breaks down precisely when $x - it$ takes on a value corresponding to a pole of the function $(1+z^2)^{-1}$. In the next subsection we shall show how surface tension prevents the formation of such singularities.

Finally we assume the fluids to be moving at the same speed and let the initial profile be given by $\eta_0(x) = (1-x^2)^3$, $|x| \leqslant 1$ and $\eta_0(x) = 0$ otherwise. Now by (8)

$$\eta(x,t) = \begin{cases} \epsilon[\{1-(x-t)^2\}^3 - 6t(x-t)\{1-(x-t)^2\}^2], & |x-t| \leqslant 1; \\ 0, \quad |x-t| > 1. \end{cases}$$

As the disturbance is initially confined to $|x| \leqslant 1$ and as its speed is that of the streams the disturbance at later times is confined to the region $t-1 \leqslant x \leqslant t+1$. Asymptotically the shape is determined primarily by the initial slope distribution.

The above examples show the extra information that one obtains by solving the initial-value problem. The modal solution determines the growth rate of each mode, which is either exponential or linear ($\alpha = 1$) in time. However, the evolution of a given initial disturbance depends on its spectral composition and the resulting superposition leads to rates of growth (for example of the peak amplitude) which depend crucially on the initial shape. Thus, all other factors being the same, two different initial disturbances can have totally different growth patterns although the basic modes are the same. Thus while modal analysis does indicate stability or instability only a solution of the initial-value problem can determine the actual growth rate of a given disturbance.

3.3. *The effects of capillarity*

We have so far assumed the interface to be free of surface tension effects. One consequence was that it was possible for the interface to develop a discontinuity after a finite time. It is to be expected that, since surface tension always makes extremely short ripples stable, its inclusion in the analysis should prevent the formation of singularities. We shall now show that this is indeed so.

The effect of capillarity is only felt through the modification of the interface boundary condition (3.3). The dynamical condition at the interface, in its linearized form, is now (Lamb 1932)

$$p_2' - p_1' = -T\frac{\partial^2 \eta'}{\partial x'^2}, \tag{9}$$

where T is the surface tension and the primes indicate dimensional quantities. This expression combined with Bernoulli's equations leads to the proper replacement, in dimensionless form, for (3.3):

$$\phi_{1t} + \phi_{1x} + \eta - \beta\{\phi_{2t} + \alpha\phi_{2x} + \eta\} = -\sigma\eta_{xx}. \tag{10}$$

Here $\sigma = gT/\rho_1 U_1^4$ is a dimensionless surface-tension parameter. Carrying through the analysis as before, one finds that the Fourier transform of the interface displacement is now given by

$$\tilde{\eta}(k,t) = \tfrac{1}{2}\epsilon\tilde{\eta}_0[\{1 - ik(1+\alpha\beta)[\beta k^2(1-\alpha)^2 + (1+\beta)(1-\beta-\sigma k^2)|k|]^{-\frac{1}{2}}\}e^{p_1(k)t} + \{1 + ik(1+\alpha\beta)[\beta k^2(1-\alpha)^2 + (1+\beta)(1-\beta-\sigma k^2)|k|]^{-\frac{1}{2}}\}e^{p_2(k)t}], \tag{11.1}$$

where the coefficients in the exponents are given by

$$p_{1,2}(k) = \frac{ik(1+\alpha\beta) \pm [\beta k^2(1-\alpha)^2 + (1+\beta)(1-\beta-\sigma k^2)\,|k|]^{\frac{1}{2}}}{1+\beta}. \tag{11.2}$$

Examining (11.2), it is clear that since σ is positive $p_{1,2}$ are purely imaginary for all sufficiently large k. Thus, although the low-wavenumber modes may be unstable, the integral (6) will now converge provided $\tilde{\eta}_0(k)$ is absolutely integrable. As a consequence the initial-value problem will now have a solution for all time, irrespective of the values of α and β, under the mild restriction on $\tilde{\eta}_0(k)$. Another consequence is that the case $\alpha = \beta = 1$ does not lead to a degeneracy and hence in this case the individual modes are neutrally stable. An unfortunate result, however, is that now the integral (6) does not lead to a simple closed-form expression even in the case $\beta = 1$.

We now reconsider the example discussed in § 3.2 which led to a singularity at $t = 1$. Let $\beta = 1$, $\alpha = -1$ and $\eta_0(x) = (1+x^2)^{-1}$. It easily follows from (11.1), (11.2) and (6) that the interface evolution is now given by

$$\eta(x,t) = \frac{\epsilon}{2}\int_0^\infty e^{-k}[\exp[kt(1-\tfrac{1}{2}\sigma k)^{\frac{1}{2}}] + \exp[-kt(1-\tfrac{1}{2}\sigma k)^{\frac{1}{2}}]]\cos kx\,dk. \tag{12}$$

While it does not appear possible to evaluate the integral in closed form, the asymptotic behaviour of $\eta(x,t)$ for large time can be found by Laplace's method. We thus find that, provided $\cos(4x/3\sigma) \neq 0$,

$$\eta(x,t) \sim \epsilon\cos\left(\frac{4x}{3\sigma}\right)\left\{\frac{2\pi}{3\sqrt{3}\,\sigma t}\right\}^{\frac{1}{2}}\exp\left[\frac{4}{3\sigma}\left(\frac{t}{\sqrt{3}}-1\right)\right] \quad \text{as} \quad t\to\infty. \tag{13}$$

The growth of the interface in the absence of surface tension and its growth when surface tension is present are compared in figure 3. It is clear that for small times the solutions are very similar, though surface tension does inhibit the growth of the interface. The major effect is, however, that the singularity at $t = 1$ is no longer present, as all the high-wavenumber modes are now stable. On the other hand the unstable low-wavenumber modes do lead to a growth of the interface. It is interesting to note that, whereas one had an algebraic singularity at $t = 1$ when $\sigma = 0$, the growth is now exponential when $\sigma \neq 0$.

4. The vortex sheet shed by a flat plate

We now consider the evolution of a vortex sheet shed by a semi-infinite flat plate. In order to simplify the analysis it will be assumed that the two streams have the same density ($\beta = 1$) and that surface tension is absent. The initial-value problem now corresponds to the situation where the right half of a rigid plate, which is flat for $x < 0$, suddenly disappears at the initial instant.

The analysis of this section is related to the paper by Orszag & Crow (1970) on the instability of a vortex sheet leaving a semi-infinite flat plate. Assuming a solution harmonic in time, Orszag & Crow determined the spatially growing instability modes using the Wiener–Hopf technique. Here we consider the corresponding initial-value problem. It is to be noted in the modal approach that, while one can easily consider either spatial or temporal harmonic modes in the absence of the plate, one is forced in an analysis including the plate to consider time-harmonic solutions (because solutions with a pure $\exp(ikx)$ behaviour are not permitted in the presence of the plate). However, for the initial-value problem, i.e. one in which the vortex-sheet displacement is prescribed, time-harmonic modes are not very useful as they cannot, in any obvious way, be combined to represent the given initial vortex-sheet displacement. It is for this reason that in the present analysis the initial-value problem is solved directly using operational methods.

4.1. *Formal solution using parabolic co-ordinates*

The standard procedure for solving the mixed boundary-value problem under consideration is the Wiener–Hopf method. All the references quoted use the method and the temptation is to closely follow the analysis of Orszag & Crow after a Laplace transform in time. However, it is soon found that an additive decomposition is required which appears difficult to perform explicitly. Here we follow an alternative, more elementary, route which leads to an explicit formula for the vortex-sheet displacement.

Let ξ and μ be parabolic co-ordinates defined by $\xi + i\mu = (x+iy)^{\frac{1}{2}}$ or

$$\left.\begin{aligned} \xi &= \{\tfrac{1}{2}(x+(x^2+y^2)^{\frac{1}{2}})\}^{\frac{1}{2}}, \\ \mu &= \pm\{\tfrac{1}{2}(-x+(x^2+y^2)^{\frac{1}{2}})\}^{\frac{1}{2}}. \end{aligned}\right\} \tag{14}$$

Now the plate maps to the line $\xi = 0$ while the nominal vortex sheet position, $y = 0$, $x > 0$, maps to $\mu = 0$; the fluid regions 1 and 2 map to the upper and lower right half-planes respectively. Since the mapping is by an analytic function, equations (1) are invariant under the transformation; the boundary conditions (2) lead to decay conditions for $|\mu| \to \infty$. The interface conditions (3) now read

$$\left.\begin{aligned} 2\xi\eta_t + \eta_\xi &= \phi_{1\mu} \\ 2\xi\eta_t + \alpha\eta_\xi &= \phi_{2\mu} \\ 2\xi\phi_{1t} + \phi_{1\xi} &= 2\xi\phi_{2t} + \alpha\phi_{2\xi} \end{aligned}\right\} \quad \text{on} \quad \mu = 0, \tag{15}$$

while the conditions on the plate lead to

$$\left.\begin{aligned} \eta(\xi, t) &= 0 \\ \phi_{1\xi} = \phi_{2\xi} &= 0 \end{aligned}\right\} \quad \text{on} \quad \xi = 0. \tag{16}$$

One notes that the interface conditions have become more complicated since the coefficients in the equations are now no longer constants. Further, the present procedure is likely to be of little use in the acoustic case as the wave equation is not invariant under the map $(x, y) \to (\xi, \mu)$.

Let $\hat{\eta}(\xi, p)$, $\hat{\phi}_1(\xi, \mu, p)$ and $\hat{\phi}_2(\xi, \mu, p)$ be the Laplace transforms with respect to time of $\eta(\xi, t)$, $\phi_1(\xi, \mu, t)$ and $\phi_2(\xi, \mu, t)$ respectively. Then the boundary conditions (15) on $\mu = 0$ transform to

$$2\xi\{p\hat{\eta} - \epsilon\eta_0(\xi)\} + \hat{\eta}_\xi = \hat{\phi}_{1\mu}, \tag{17.1}$$

$$2\xi\{p\hat{\eta} - \epsilon\eta_0(\xi)\} + \alpha\hat{\eta}_\xi = \hat{\phi}_{2\mu}, \tag{17.2}$$

$$2\xi\{p\hat{\phi}_1 - \phi_1(\xi, 0, 0)\} + \hat{\phi}_{1\xi} = 2\xi\{p\hat{\phi}_2 - \phi_2(\xi, 0, 0)\} + \alpha\hat{\phi}_{2\xi}. \tag{17.3}$$

In the above $\eta_0(\xi)$ is taken to mean the expression resulting from replacing x by ξ^2 everywhere in the expression for $\eta_0(x)$ (strictly, one should use the notation $\eta_0(\xi^2)$).

Now inspection of the non-constant coefficient equations (17) suggests that some advantage might be gained by using full-range Fourier transforms in ξ. Since the physical problem has so far been defined only in the right half-plane $\xi \geqslant 0$, we define

$$\phi_{1,2}(-\xi,\mu,t) = \phi_{1,2}(\xi,\mu,t), \tag{18.1}$$

$$\eta(-\xi,t) = -\eta(\xi,t), \tag{18.2}$$

thereby extending the solution domain to the whole of the (ξ,μ) plane. As the definitions (18.1) and (18.2) force $\phi_{1,2}$ to be symmetric and η to be anti-symmetric, it follows that, provided $\phi_{1,2}$ are continuously differentiable in ξ and $\eta(\xi)$ is continuous in ξ, the boundary conditions (16) will be automatically satisfied. It is easy to verify that Laplace's equation and the boundary conditions (15) are invariant under $\xi \rightarrow -\xi$ with the definitions (18). Thus the definitions (18) are consistent and we can proceed to use full-range Fourier transforms in ξ. Let us define

$$C(k,p) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} \hat{\eta}(\xi,p)\, e^{ik\xi}\, d\xi \tag{19}$$

with similar definitions for the potentials.

One proceeds now by Fourier transforming the boundary conditions (17) and solving the resulting non-constant coefficient, inhomogeneous, ordinary differential equation for $C(k,p)$ (the details may be found in Shankar 1980). We thus find

$$C(k,p) = \begin{cases} \dfrac{\epsilon\tilde{\eta}_0}{p} + \dfrac{(1+\alpha^2)\, i\epsilon}{4(1-\alpha)\, p^2} \displaystyle\int_0^k \{e^{\delta(k^2-\zeta^2)} - e^{\delta'(k^2-\zeta^2)}\}\, \zeta\tilde{\eta}_0(\zeta)\, d\zeta \quad (\alpha \neq 1), & (20.1) \\[2ex] \dfrac{\epsilon\tilde{\eta}_0}{p} - \dfrac{\epsilon}{8p^2} \displaystyle\int_0^k \zeta(k^2-\zeta^2)\, \tilde{\eta}_0(\zeta)\, e^{-(k^2-\zeta^2)/4p}\, d\zeta \quad (\alpha = 1), & (20.2) \end{cases}$$

where δ and δ' are defined by

$$\delta, \delta' = \{-(1+\alpha) \pm (1-\alpha)\, i\}/8p. \tag{21}$$

The vortex sheet displacement can now be recovered by the inversion formula

$$\eta(\xi,t) = \frac{1}{2\pi i}\frac{1}{\sqrt{(2\pi)}} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{pt}\, dp \int_{-\infty}^{\infty} e^{-ik\xi} C(k,p)\, dk. \tag{22}$$

This completes the formal solution of the problem, whenever it exists, in the sense that $\eta(\xi,t)$ can be determined given $\epsilon\eta_0(x)$. The time inversion in (22) can readily be done and one finds, when $\alpha \neq 1$,

$$\eta(\xi,t) = \epsilon\eta_0(\xi) + \frac{\epsilon}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} \left[\frac{(1+\alpha^2)\, i}{4(1-\alpha)} \int_0^k \left\{ \frac{\sqrt{t}\, I_1([(-(1+\alpha)+(1-\alpha)\, i)\,(k^2-\zeta^2)\, t/2]^{\frac{1}{2}})}{[(-(1+\alpha)+(1-\alpha)\, i)\,(k^2-\zeta^2)/8]^{\frac{1}{2}}} - \frac{\sqrt{t}\, I_1([(-(1+\alpha)-(1-\alpha)\, i)\,(k^2-\zeta^2)\, t/2]^{\frac{1}{2}})}{[(-(1+\alpha)-(1-\alpha)\, i)\,(k^2-\zeta^2)/8]^{\frac{1}{2}}} \right\} \zeta\tilde{\eta}_0(\zeta)\, d\zeta \right] e^{-ik\xi}\, dk. \tag{23}$$

When $\alpha = 1$, the time inversion leads to the formula

$$\eta(\xi,t) = \epsilon\eta_0(\xi) + \frac{\epsilon i t}{\sqrt{(2\pi)}} \int_0^{\infty} \left[\int_0^k J_2([(k^2-\zeta^2)\, t]^{\frac{1}{2}})\, \zeta\tilde{\eta}_0(\zeta)\, d\zeta \right] \sin k\xi\, dk. \tag{24}$$

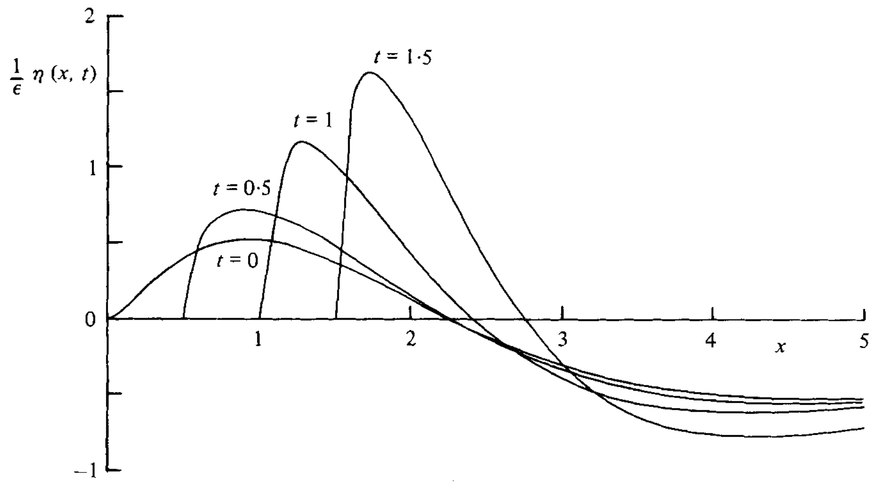

FIGURE 4. The evolution of the vortex sheet when $\alpha = \beta = 1$ and $\eta_0(x) = 2x^{\frac{3}{2}}/\sqrt{\pi} + (2x^2 - x)\, i\,\mathrm{erf}\,(i\sqrt{x})\, e^{-x}$.

4.2. *An example*

It is clear from formulae (20)–(24) that, in general, detailed calculations would be involved if one wished to compute the evolution of any specific initial profile $\eta_0(x)$. We wish now to give an example for which most of the integrals in question can be evaluated in closed form. It turns out that this example can be solved trivially by other means but it does give one confidence in the final formulae obtained in § 4.1.

Let us for analytical convenience assume both fluids to move at the same speed, i.e. $\alpha = 1$, and let the initial vortex sheet shape be given by

$$\eta_0(x) = 2x^{\frac{3}{2}}/\sqrt{\pi} + (2x^2 - x)\, i\,\mathrm{erf}\,(i\sqrt{x})\, e^{-x}. \tag{25}$$

It turns out (see Shankar 1980) that for this profile the integral over ζ in (20) can be evaluated in closed form, the Fourier inversions over k are standard, and the Laplace inversion over p can be considerably simplified to finally yield

$$\eta(x,t) = \frac{\epsilon H(\xi^2 - t)}{2\pi}\int_0^\infty \mathrm{Im}\,[\{(-2\nu^2 + 20\nu^3 - 24\nu^4)\,\rho^{-\frac{5}{2}}\,e^{-\frac{5}{4}\pi i} + (-\nu + 5\nu^2 - 4\nu^3)\,(-\rho^{-\frac{3}{2}}\,e^{-\frac{3}{4}\pi i} + 2\xi^2\rho^{-\frac{1}{2}}\,e^{-\frac{1}{4}\pi i})\}\,\{\mathrm{erf}\,(i\xi\sqrt{\rho}\,e^{\frac{1}{4}i\pi})\,e^{i\rho(t-\xi^2)}\}]\,d\rho, \tag{26}$$

where $\nu = \rho/(\rho + i)$. Figure 4 shows the evolution of the vortex sheet in this case as given by (26). Examining figure 4, it now becomes obvious† that the special choice of (25) and all the calculation leading to (26) is unnecessary! Since both fluids move at the same speed the disturbance (vorticity) gets carried by the fluids at their speed while it grows as a result of the instability. This results, since the plate is flat upstream of the trailing edge, in the vortex sheet being flat up to a distance t from the trailing edge. It follows that, since the plate does not affect the straight tail of the disturbance,

† The author is most indebted to a referee for having made this fact 'obvious' to him by giving an ingeniously simple reasoning, based on vorticity distribution arguments, to show that (8) holds even in the semi-infinite vortex-sheet case.

formula (8) holds even for the semi-infinite vortex sheet. This simplification is possible only in the case $\alpha = 1$. We note also that (22) and (24) are consequently integral representations of (8), although this fact would be hard to prove directly.

5. Conclusion

We have in this paper discussed the initial-value problem for the vortex sheet separating two uniform, incompressible fluid regions. It would appear that the linear theory considered here would be physically valid at least for combinations of sufficiently small initial amplitudes and sufficiently small time intervals. Even where the present solution breaks down after a finite time interval, it should be a guide for sufficiently small times to the solutions that might be obtained with the inclusion of the effects of surface tension and viscosity. In any case, the simple formulae (7) and (8) should be of interest as exact solutions to an initial-value problem in vortex-sheet dynamics.

It has been shown here that capillarity does indeed suppress the tendency for the vortex sheet to develop singularities. Viscosity should have a similar effect and it would be most interesting to analyse its effects. The latter problem, however, poses serious analytical difficulties. One would no longer be able to use a potential, the linearized equations would involve variable coefficients, the orders of the equations would increase and correspondingly the boundary conditions will be more complex. If these difficulties are overcome, the resulting solution would provide a most valuable extension to the present results.

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